

Comment on Ghang and Nowak's "Indirect reciprocity with optional interactions" *

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1 Introduction

The purpose of this note is to point out some simple mathematical errors in the article "Indirect reciprocity with optional interactions" (*Journal of Theoretical Biology* 365 (2015) 1–11) by Whan Ghang and Martin Nowak, which affect their results and conclusions. We note various ways that these errors could be fixed, but all require fundamental revisions of their model and lead to counterintuitive conclusions. We also discuss their use of a Nash equilibrium (as opposed to an evolutionary stability) approach in deriving conditions for the evolution of cooperation with indirect reciprocity, and mention some precedents for their model which have appeared in the biology and economics literature.

2 The Ghang & Nowak Game

Ghang and Nowak study an "optional game" in a population of N individuals in which in each round a prisoner's dilemma game is played between two randomly matched players. When two players are matched, a game takes place only if both accept to play. The repeated game continues with probability w after each round, so the average number of rounds is $M = 1 + w + w^2 \dots = \frac{1}{1-w}$ and the average number of potential games offered to each player is $h = \frac{2}{N}M$, where $\frac{2}{N}$ is the matching probability. Throughout the paper they consider the case $M > 1$ (i.e. $w > 0$).

Ghang and Nowak specify two "types" of player, "cooperators" and "defectors". Cooperators pay a cost, c , for the other player to receive a benefit,

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b , where $b > c > 0$. Defectors pay no cost and provide no benefit. They assume that a cooperator accepts to play a game as long as the other player is unknown or known to be a cooperator. A defector always accepts to play the game. If a game occurs where one individual is a cooperator and the other is a defector, then the reputation of the defector is established in the population with probability Q . If a player is known to be a defector, then in all subsequent rounds cooperators refuse to play with him. A game between two defectors leads to zero payoff for both players, and no reputation is established or revealed.

Ghang and Nowak present a "Nash equilibrium type argument" by which they appear to mean that they assume that players choose a type ("cooperator" or "defector") prior to the game beginning, and ask whether all players choosing to be cooperators constitutes a Nash equilibrium. The average payoff of a cooperator in the repeated game if everyone cooperates is denoted by F_N and the average payoff of a single defector by G_{N-1} . It is then easy to see that

$$F_N = h[b - c],$$

which is Ghang and Nowak's equation (1), and with some derivation that

$$G_{N-1} = \frac{hb}{1 + hwQ},$$

which is equivalent to their equation (2). The condition for all players choosing to be the cooperative type at the beginning of the game to be a strict Nash equilibrium, $F_N > G_{N-1}$, they derive as

$$h > \frac{1}{BQ} + \frac{2}{N}, \quad (3)$$

in their expression (3), where $B = \frac{b-c}{c}$.¹ An equivalent way to write Ghang and Nowak's condition (3) is

$$h > \frac{1}{BwQ},$$

since $h = \frac{2}{N}M$, or

$$hBwQ > 1.$$

In discussion of their expression (3) Ghang and Nowak claim that the "additive" term $\frac{2}{N}$ becomes small for large population size and hence in the limit they obtain the simple condition for cooperation to be a strict

¹Note that, more generally, the condition for a player to prefer to choose to be a cooperative type when there are $N-k-1$ other cooperative types (i.e. k defectors) is given by $F_{N-k} > G_{N-k-1}$.

Nash equilibrium: $hBQ > 1$ (Ghang and Nowak's expression (4)). This is incorrect, as the authors seem to consistently ignore the fact that $h = \frac{2}{N}M$ and also goes to 0 in the limit as population size increases. Indeed, since h contains the matching probability $\frac{2}{N}$, cooperation cannot be sustained in large populations in their model. Another way of saying this is that as N increases, the probability of future matches decreases, and the incentive for any matched player to cheat increases until defection is the only equilibrium (and ultimately dominant) strategy. The game becomes effectively a one shot prisoners' dilemma for all players.

For cooperation to be sustained as a Nash equilibrium in this model the population size N must satisfy

$$N < \frac{2BwQ}{1-w}.$$

N can only grow arbitrarily large if the continuation probability w approaches 1 ($M \rightarrow \infty$), so that all of the infinitely patient players are assured of playing a large, and in the limit infinite, number of future games in which their reputations will matter.

This error feeds through into the remainder of Ghang and Nowak's analysis: in their discussion of (9) and the claim that (9) converges to (4) for large N ; in section 2.1 when they consider "game dynamics" for "large population size" (the LHS of their inequality in (11), for example, converges to $\log 1 = 0$ as the population size increases and (11) can't be satisfied for arbitrarily large populations); and whenever they derive conditions involving "large N " in subsequent sections 4 and 5, and in their Conclusion.

One way that this problem could be addressed would be to simply fix the average number of potential games offered to each player, h , an assumption Ghang and Nowak appear to adopt in Section 2.2 of their paper when they study deterministic evolutionary dynamics in the limit $N \rightarrow \infty$, and treat h as exogenous. Since $h = 2M/N$, fixing h while leaving the rest of the model essentially unchanged would make M , the average number of rounds, and hence w , the probability that the game continues after any given round, endogenous functions of N . Making the continuation probability of the game a function of population size would appear to be difficult to justify, however, and doing so leads to the counterintuitive result that cooperation becomes easier to sustain as population size N grows larger (a result already implied by Ghang and Nowak's incorrect derivation of their expression (4) from (3)).

Perhaps a more natural way of preventing the probability of future matches from going to zero as N grows large, would be to introduce a matching probability that is independent of population size. A matching probability p , independent of N , fixes h in a natural way by $h = pM$. This would fundamentally alter the model – making the population size irrelevant for the likelihood of matching. It also implies that the equilibrium conditions for sustaining cooperation become independent of N , while intuitively,

coordinating on the cooperative strategy and keeping track of other players' reputations should become harder as the population size grows larger.

3 Coalitions and Stability: Nash Equilibrium or Finite Population ESS?

It is also worth pointing out that to ask, as Ghang and Nowak do in deriving their expression (10), "whether cooperation is stable against a coalition of k players who switch simultaneously to defection" is unusual in a noncooperative game framework, if not entirely meaningless. There is typically no place in Nash equilibrium, non-cooperative game theory for such a cooperative idea - it usually does not make sense to consider simultaneous deviations by a group of players, since there is no way to coordinate such "out-of-equilibrium" play.²

In any case, the result of their expression (10) that the "condition for cooperation to be a strict Nash equilibrium (3) implies stability against a coalition of k players switching to defection" is obvious and does not require any elaborate calculations. If a single defector reduces his or her payoff when playing against a population of cooperators (as implied by (3)), then multiple defectors must do worse still, as they will with some probability be matched against each other, and such matches result in zero payoffs for both players. That is, the condition $F_N > G_N - k$ for $k > 1$ follows directly from $F_N > G_N - 1$, since the payoff to defectors is smaller the more defectors there are.

It is in any event unclear what the relevance of the authors' Nash equilibrium analysis is, when they repeatedly refer to their paper as establishing "the conditions for evolution of cooperation" rather than the conditions for cooperation to be a Nash equilibrium between rational human actors. If they do literally intend a Nash equilibrium analysis, their model makes limited sense, as they assume that the players are "types" rather than rational actors choosing strategies, and it is unclear how a rational player could, or would wish to, choose to be a "type" (i.e. either a "cooperator" or a "defector") for the entire history of the game.

If an evolutionary analysis is intended however, traditionally one would look for evolutionary stable strategies (ESS).³ In this case, the authors

²If we are to allow coalitions of k players to form and choose strategies (i.e. types), we should also allow for the coalition of all N players to form and choose cooperative strategies. The condition for cooperation to be a strict Nash equilibrium implies that cooperation is in the core of the cooperative game. Alternatively, global cooperation (but not defection) is "coalition proof" in the sense of Bernheim, Peleg and Whinston (1987) or a "strong equilibrium" in the sense of Aumann (1959). See Fudenberg and Tirole (1991), p. 22.

³See Ania (2008), Nowak *et al.* (2004), Schaffer (1988) and Tanaka (200) for definitions of evolutionary stable strategies in finite population models.

should have compared F_{N-1} with G_{N-1} for finite populations in deriving their expression (3) (they do perform this calculation to derive their condition (9)), and $F_{N-k} > G_{N-k}$ in (10). The condition for non-invasion of a putative ESS by a group of size k is that the payoffs of those continuing to play the cooperative strategy, $F_N - k$, is greater than the payoffs of the k deviators, $G_N - k$. For a k for which $G_N - k > F_N - k$, the deviators achieve higher payoffs than the cooperators, reproduce more rapidly and take over the population. Ghang and Nowak's derivations of (5) through (9) seem to be intended to provide precisely the information needed for an ESS analysis.⁴

4 Precedents for the Ghang & Nowak Model

Finally, it is doubtful that Ghang and Nowak's claim for originality in proposing a "new" model of indirect reciprocity with optional interactions (or partner choice) is entirely accurate. Kitcher (1993) and Batali and Kitcher (1995) analyze optional PD games in a similar set up to that used by the authors, and allow for a richer set of strategies, although they consider only direct (as opposed to indirect) reciprocity (see also Szabo and Hauert 2002). Greif (1993) (also Greif 2006 and Harbord 2006) uses a model of optional interactions in a one-sided prisoners' dilemma in his analysis of cooperative trading relationships and indirect reciprocity between the medieval Maghribi merchants. Greif's analysis has the merit of showing why, in equilibrium, players will follow the social norm of refusing to engage (or play with) an agent who has ever cheated (or defected) in the past, making indirect reciprocity self enforcing without the need to invoke "multi-level" or "higher-order" punishments.

Since Ghang and Nowak consider a "types" model, they are unable to ask what incentives players have to follow the social norm and refuse to play stage games with players with a reputation for defection.⁵ If the equilibrium incentives are for cooperation, then a rational actor who has defected in the past should cooperate in all future interactions, so why should a player give up a potentially profitable opportunity which may not recur by following the social norm and refusing to play with him or her? The reason for doing so is straightforward and reveals the power of this simple social norm in

⁴In this case it is not true that "*the condition for cooperation to be a strict Nash equilibrium (3) implies stability against a coalition of k players switching to defection.*" If $k = N - 1$ for example, then Ghang and Nowak have told us (just below their expression (8)) that "*a single cooperator in a population of defectors always has a lower average payoff than the defectors*". A defector's average payoffs always weakly exceeds 0. There is a critical value of k however, for which cooperators' average payoffs become negative.

⁵Ghang and Nowak refer to this as the "action rule" and use the term "social norm" in a nonstandard way. We are using the well-established definition of a social norm as found in the economics and game theory literature. See, for instance, Peyton Young (2008), Kandori (1992) and Okuno-Fujiwara and Postlewaite (1995).

making cooperation feasible. When a player with a past reputation for defection is paired with an cooperative player, he (or she) does not expect to encounter any further opportunities for profitable play given that all of the other players are following the social norm. Hence his (her) incentive is to cheat at every opportunity. Knowing this, a cooperative player will refuse to play with a player with a prior reputation for cheating, and the social norm is self-enforcing (i.e. a Nash equilibrium).

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